

CALCULUS II TUTORIAL: SOLUTIONS (5)

① $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} \dots \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

$$\int_0^x \tan x \, dx = \int_0^x \frac{(\sin x) \, dx}{\cos x} = - \int_0^x \frac{d(\cos x)}{\cos x} = -\ln|\cos x| \Big|_0^x = \ln|\sec x|$$

where lower limit is zero since $\ln|\sec 0| = \ln 1 = 0$.

Integrating original series term by term gives

$$\int_0^x \tan x \, dx = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175} + \dots$$

This converges to $\ln|\sec x|$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (same range as $\tan x$ series). Note $\ln|\sec \frac{\pi}{2}| = \ln \infty = \infty$.

Differentiating original series term by term gives

$$\sec^2 x = \frac{d}{dx}(\tan x) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} \dots$$

for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Note $\sec^2 \frac{\pi}{2} = \infty$.

② Start with $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \quad (-1 < x < 1)$.

Differentiating: $\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2} = 1 + 2x + \dots + nx^{n-1} + \dots$

$$\Rightarrow x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2} = x + 2x^2 + \dots + nx^n + \dots$$

Differentiating again: $\frac{d}{dx} \left\{ \frac{x}{(1-x)^2} \right\} = 1 + 4x + \dots + n^2 x^{n-1} + \dots$

$$\Rightarrow x \frac{d}{dx} \left\{ \frac{x}{(1-x)^2} \right\} = x + 4x^2 + \dots + n^2 x^n + \dots = \sum_{n=1}^{\infty} (n^2 x^n)$$

But LHS = $x \left\{ \frac{(1-x)^2 + x \cdot 2(1-x)}{(1-x)^4} \right\} = \frac{x(1+x)}{(1-x)^3}$

Putting $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} \right) = \frac{\frac{1}{2} \cdot \frac{3}{2}}{\left(\frac{1}{2}\right)^3} = 6$. NB. this works for any $x \in (-1, 1)$

③ Whenever sum less than $\frac{1}{3}$, add positive

terms in order until it goes above $\frac{1}{3}$. Then add negative terms in order until it goes below $\frac{1}{3}$ etc. Thus $S_n \rightarrow \frac{1}{3}$.

Eg. $1 - \frac{1}{2} \left[\frac{1}{2} \right] - \frac{1}{4} \left[\frac{1}{4} \right] + \frac{1}{3} \left[\frac{7}{12} \right] - \frac{1}{6} \left[\frac{5}{12} \right] - \frac{1}{8} \left[\frac{7}{24} \right] + \frac{1}{5} \left[\frac{59}{120} \right] - \frac{1}{10} \left[\frac{47}{120} \right]$

$25 = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

collecting terms with same denominators. Hence $25 = 5!$

$$(3) (a) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1} \stackrel{u_n}{=} \frac{1}{2} - \frac{2}{9} + \frac{3}{28} - \dots \text{ (alternating series)}$$

$$u_n > 0, u_n > u_{n+1}, u_n \approx \frac{1}{n^2} \rightarrow 0 \Rightarrow \text{convergence.}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which converges (p-series with } p=2)$$

Hence series converges absolutely.

$$(b) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2} = 2 - \frac{3}{4} + \frac{4}{9} - \dots \text{ converges since}$$

$$u_n = \frac{1+n}{n^2} \rightarrow \frac{1}{n} \rightarrow 0, u_n > 0, u_n > u_{n+1} \text{ for all } n.$$

$$\sum_{n=1}^{\infty} \frac{1+n}{n^2} > \sum_{n=1}^{\infty} \left(\frac{n}{n^2} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges.}$$

Hence series converges conditionally.

$$(c) \sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) = -(\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) - (2-\sqrt{3}) + \dots$$

$$\text{For large } n: \left[\text{or use } u_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0 \right]$$

$$u_n = \sqrt{n+1} - \sqrt{n} = \sqrt{n} \left(1 + \frac{1}{n} \right)^{\frac{1}{2}} - \sqrt{n} \approx \frac{\sqrt{n}}{2n} = \frac{1}{2\sqrt{n}} \rightarrow 0.$$

$$\text{So } u_n \rightarrow 0, u_n > 0, u_n > u_{n+1} \Rightarrow \text{convergence}$$

$$\text{But } \sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n}) \approx \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (p-series with } p=\frac{1}{2})$$

Hence series converges conditionally, $\leftarrow$ geometric

$$(4) (a) \sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n} = 1 + \left(\frac{x-2}{10} \right) + \left(\frac{x-2}{10} \right)^2 + \dots \rightarrow \frac{1}{1 - \frac{x-2}{10}} = \frac{10}{12-x}$$

$$\text{Converges for } \left| \frac{x-2}{10} \right| < 1 \Rightarrow -1 < \frac{x-2}{10} < 1 \Rightarrow -8 < x < 12 \leftarrow \text{(interval of cge)}$$

Diverges elsewhere. Radius of cge = 10. No region of cond' cge.

$$(b) \sum_{n=0}^{\infty} \frac{x^n}{n\sqrt{n} 3^n} \Rightarrow \left| \frac{u_{n+1}}{u_n} \right| = \frac{n\sqrt{n}|x|}{(n+1)\sqrt{n+1} 3} \rightarrow \frac{|x|}{3}.$$

$$\text{Hence converges absolutely for } |x| < 3 \Rightarrow -3 < x < 3$$

$$x = 3 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}} \text{ converges (p-series with } p=\frac{3}{2})$$

$$x = -3 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{n\sqrt{n}} \text{ converges since } u_n > 0, u_n > u_{n+1}, u_n \rightarrow 0;$$

Hence interval of convergence is $-3 \leq x \leq 3$. No cond' cge region

$$(c) \sum_{n=1}^{\infty} n^n x^n \Rightarrow \left| \frac{u_{n+1}}{u_n} \right| = \frac{(n+1)^{n+1} x}{n^n} = \left(\frac{n+1}{n} \right)^n n |x| \rightarrow \infty$$

unless $|x| = 0$. Hence series diverges unless $x = 0$.

Radius of cge = 0. No region of conditional cge.